

ELECTRIC-MAGNETIC DUALITY IN NONCOMMUTATIVE GEOMETRY

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ABSTRACT

The structure of S-duality in $U(1)$ gauge theory on a 4-manifold M is examined using the formalism of noncommutative geometry. A noncommutative space is constructed from the algebra of Wilson-'t Hooft line operators which encodes both the ordinary geometry of M and its infinite-dimensional loop space geometry. S-duality is shown to act as an inner automorphism of the algebra and arises as a consequence of the existence of two independent Dirac operators associated with the spaces of self-dual and anti-selfdual 2-forms on M . The relations with the noncommutative geometry of string theory and T-duality are also discussed.

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†Work supported in part by the Natural Sciences and Engineering Research Council of Canada.

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One of the oldest forms of duality in physics is electric-magnetic duality which asserts the invariance of the vacuum Maxwell equations under the interchange of electric and magnetic fields. This phenomenon is an example of an ‘S-duality’ which originally appeared in the context of four-dimensional abelian gauge theories [1], such as electrodynamics with instanton term[‡]

$$S[A] = \frac{1}{16\pi} \int_M d^4x \left(-\frac{4\pi}{g^2} F_{\mu\nu} F^{\mu\nu} + \frac{i\theta}{2\pi} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} \right) \quad (1)$$

defined on a four-dimensional Euclidean spacetime manifold M , where g is the electro-magnetic coupling constant and θ is the vacuum angle. S-duality in this case implies the invariance of the quantum field theory under the modular transformation $\tau \rightarrow -1/\tau$ of the parameter

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \quad (2)$$

which lives in the upper complex half-plane. It therefore relates the strong and weak coupling regimes of the quantum field theory. This duality extends to $N = 4$ supersymmetric Yang-Mills theory [2], and recently this property has been exploited to obtain non-perturbative solutions of $N = 2$ supersymmetric quantum chromodynamics [3] leading to a picture of quark confinement in terms of the condensation of magnetic monopoles. S-duality has also enabled much progress towards the exact solution of $N = 1$ supersymmetric gauge theories [4].

Many generic features of S-duality can be learned from the simple quantum field theory (1). The global duality properties of this model have been studied from a path integral point of view in [5] and in terms of canonical transformations on the gauge theory phase space in [6]. In this letter we will show how the duality properties of the theory (1) naturally emerge within the formalism of noncommutative geometry [7]. We will show that the same features of the noncommutative spacetime formulation of target space duality in string theory [8] emerge in this description. This extends the worldsheet formulations of the noncommutative geometry of string spacetimes [8]–[11] to a genuine spacetime description. These constructions could have strong implications on the unified structure of M Theory [12] which relates different spacetime theories to one another via target space and S-duality transformations. The low-energy dynamics of M Theory has been recently conjectured to be described by a dimensionally-reduced supersymmetric Yang-Mills theory [13]. A noncommutative geometry formalism for this description has been discussed in [14] (see also [11]).

The basic object which describes a metric space in noncommutative geometry is the spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where \mathcal{A} is a $*$ -algebra of bounded operators acting on a separable Hilbert space \mathcal{H} and D is a (generalized) Dirac operator on \mathcal{H} . A spin-manifold

[‡]Here and in the following we shall not write explicit metric factors and all quantities are implicitly covariant.

M with metric $g_{\mu\nu}$ is described by the choice of $\mathcal{H} = L^2(M, S)$, the space of square-integrable spinors on M (i.e. L^2 -sections of the spin bundle $S(M)$), and the *abelian* algebra $\mathcal{A} = C^\infty(M, \mathbb{C})$ of smooth complex-valued functions on M acting by pointwise multiplication in \mathcal{H} . This is the canonical $*$ -algebra associated with any manifold, and it determines the topology and differentiable structure of a space through the smoothness criterion. In fact, there is a one-to-one correspondence between the set of all Hausdorff topological spaces and the collection of commutative C^* -algebras, and therefore the study of the properties of spacetime manifolds can be substituted by a study of the properties of abelian $*$ -algebras. The usual Dirac operator $D = ig^{\mu\nu}\gamma_\mu\nabla_\nu$ then describes the Riemannian geometry of the manifold, where the real-valued gamma-matrices obey the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ and ∇_μ is the covariant derivative constructed from the spin-connection. Thus, roughly speaking, D is the “inverse” of the infinitesimal dx which determines geodesic distances in the spacetime. It encodes the Riemannian geometry via both the gamma-matrices and the spin-connection. Note that this spectral triple naturally arises from quantizing the free geodesic motion of a test particle on M . In that case \mathcal{H} is the Hilbert space of physical states, \mathcal{A} is the algebra of observables, and the Hamiltonian $H = -D^2$ is the Laplace-Beltrami operator. One power of this description is the possible generalization to *noncommutative* algebras \mathcal{A} leading to the notion of a “noncommutative space”, such as that anticipated from string theory [8].

In the following we will construct a spectral triple appropriate for the description of the duality properties of the abelian gauge theory (1). The “space” is described by the noncommutative algebra of Wilson-’t Hooft line operators which incorporates the ordinary (commutative) spacetime geometry of the 4-manifold M , its infinite-dimensional loop space geometry, and also the geometry of the gauge theory (1). We describe the similarities between this algebra and the vertex operator algebras that are used in the construction of string spacetimes [8]–[11] and also those of M Theory [11, 14]. We shall show that the duality symmetries arise in the same way that target space duality did in [8]. Namely, the “chiral” structure of the theory, determined by the decomposition of the space of 2-forms on M into self-dual and anti-selfdual parts, implies the existence of two independent Dirac operators for the noncommutative geometry. Duality is then represented as a change of Dirac operator which is just a change of metric for the noncommutative geometry, i.e. an isometry of the “space”. Furthermore, S-duality is represented as an inner automorphism of the $*$ -algebra \mathcal{A} , meaning that in this context it is a gauge transformation representing internal fluctuations of the geometry. We also discuss the similarities between the noncommutative geometries for gauge fields and strings, illustrating the unified description of duality and the spacetime structure of string theory that emerges from the formalism of noncommutative geometry.

Before constructing the appropriate spectral triple, we shall need some basic results concerning the geometry of 4-manifolds. Consider the $U(1)$ gauge theory (1) on the 4-manifold M , which we assume is closed and admits a spin structure (equivalently it has

a vanishing second Stiefel-Whitney class). We choose a complex line bundle $L \rightarrow M$ and a gauge connection 1-form A on L with curvature F . The curvature obeys the Bianchi identity $dF = 0$ and has integer period around 2-cycles $\Sigma \subset M$, $\oint_{\Sigma} F/2\pi \in \mathbb{Z}$. Picking a conformal equivalence class of Euclidean signature metrics $g_{\mu\nu}$ on M , we can define a Hodge star-operator \star on M and consider the 2-form $\star F$ dual to the curvature F . From this we define the self-dual and anti-selfdual decompositions

$$F^{\pm} = \frac{1}{2}(F \pm \star F) \quad (3)$$

according to the splitting of the vector space of 2-forms on M as $\Omega^2(M) = \Omega_+^2(M) \oplus \Omega_-^2(M)$. The line bundle L is classified topologically by the first Chern class

$$p^{\perp} \equiv c_1(L) = [F/2\pi] \in H^2(M; \mathbb{Z}) \quad (4)$$

For classical field configurations p^{\perp} is a harmonic form, $dp^{\perp} = d\star p^{\perp} = 0$, and the closed 2-form F can be written using the Hodge decomposition as

$$F = dA + 2\pi p^{\perp} \quad (5)$$

where A is a single-valued 1-form. For a given complex line bundle L , the connections A obeying (5) live in the torus $H^1(M; \mathbb{R})/H^1(M; \mathbb{Z})$, where the real cohomology classes take into account all canonical locally gauge-equivalent connections, while the integer cohomology accounts for the equivalence under large gauge transformations.

The second cohomology group $H^2(M; \mathbb{Z})$ carries the intersection form

$$G(\alpha, \beta) = \int_M \alpha \wedge \beta \quad \text{for } \alpha, \beta \in H^2(M; \mathbb{Z}) \quad (6)$$

Any closed surface $\Sigma \subset M$ can be expressed in terms of a canonical basis Σ^a of homology 2-cycles. The inverse G^{ab} of the intersection matrix $G_{ab} \equiv G(\alpha_a, \alpha_b)$, with α_a a basis of $H^2(M; \mathbb{Z})$ dual to Σ^a , i.e. $\oint_{\Sigma^a} \alpha_b = \delta_{ab}$, is the signed intersection number $G^{ab} = \Sigma^a \cap \Sigma^b$ of the homology 2-cycles. The intersection form (6) of the 4-manifold M is always symmetric and non-degenerate, so that the matrix G_{ab} determines a quadratic form on the lattice $H^2(M; \mathbb{Z})$. Since M is a spin manifold, $H^2(M; \mathbb{Z})$ is even, and by Poincaré duality $H^2(M; \mathbb{Z})$ is self-dual. Thus the second cohomology group of the spacetime equipped with the inner product G_{ab} defines an even self-dual Lorentzian lattice $\Lambda = H^2(M; \mathbb{Z}) = H_+^2(M; \mathbb{Z}) \oplus H_-^2(M; \mathbb{Z})$. Its rank is the second Betti number $b_2 = \dim H^2(M)$ and its signature is (b_2^+, b_2^-) , with $b_2^{\pm} = \dim H_{\pm}^2(M)$ the dimensions of the spaces of self-dual and anti-selfdual harmonic 2-forms, respectively.

The Maxwell action (1) with instanton term can be written as

$$S[A] = \frac{i}{4\pi} \int_M (\bar{\tau} F^+ \wedge F^+ + \tau F^- \wedge F^-) = \frac{i}{4\pi} \int_M F \wedge \tilde{F} \quad (7)$$

where

$$\tilde{F} = \frac{\theta}{2\pi} F + \frac{4\pi i}{g^2} \star F = \tau F^+ + \bar{\tau} F^- \quad (8)$$

is the dual field strength. It is evident from (7) that F and \tilde{F} , or alternatively F^+ and F^- , play a symmetric role in the theory. The classical equations of motion are $d \star F = 0$, which combined with the Bianchi identity yield the Maxwell equations of electrodynamics. This implies that $d\tilde{F} = 0$, so that by Poincaré's lemma, there exists locally a one-form \tilde{A} on M which is a potential for the dual field strength, $\tilde{F} = d\tilde{A} + \tilde{p}^\perp$ (if M is a simply-connected 4-manifold then this is true globally). The dual gauge field \tilde{A} is related to A by

$$\tilde{A} = \frac{4\pi i}{g^2} d^{-1} \star dA + \frac{\theta}{2\pi} A + k^\perp \quad (9)$$

where k^\perp is a harmonic one-form. But other than this formal non-local expression, there is no simple relationship between A and \tilde{A} . Moreover, their simultaneous existence ceases in the presence of magnetic or electric sources. Nevertheless, when both electric and magnetic charges are present (or both absent), A and \tilde{A} are equivalent, and one could use either one as a configuration space variable. Here we will consider both of them on equal footing as independent variables in an enlarged algebra. We will then project onto a space with only one potential by imposing the physical constraints represented by Gauss' law.

To study the quantum field theory, we employ canonical quantization. For this, we work in a three-dimensional spatial slice of the 4-manifold M . Locally, in this region, the spacetime is the product manifold $M = \mathbb{R} \times M_3$, where M_3 is a compact 3-manifold without boundary, and in this region we fix the temporal gauge $A_0 = \tilde{A}_0 = 0$. Corresponding to A and \tilde{A} there are two canonically conjugate momenta

$$\Pi_i = \frac{\delta S}{\delta \partial_0 A^i} = -4 \star \tilde{F}_{0i} = 4 (\bar{\tau} F_{0i}^+ - \tau F_{0i}^-) \quad , \quad \tilde{\Pi}_i = \frac{\delta S}{\delta \partial_0 \tilde{A}^i} = 4 \star F_{0i} = 4 (F_{0i}^+ - F_{0i}^-) \quad (10)$$

with the second class constraints $\Pi_0 \sim 0, \tilde{\Pi}_0 \sim 0$, and we define $\Pi_i^\pm = 2(\tau - \bar{\tau}) F_{0i}^\pm$. The generator of local gauge transformations is given by the variation[§]

$$\mathcal{G} = \frac{\delta S}{\delta A_0} = \frac{i}{\pi} \partial^i (\bar{\tau} F_{0i}^+ - \tau F_{0i}^-) = \frac{i}{4\pi} \partial^i \Pi_i \quad (11)$$

and Gauss' law is the second class constraint $\mathcal{G} \sim 0$ to be imposed on the physical state space of the gauge theory. The Hamiltonian can be written as

$$H = \int_{M_3} d^3x \frac{1}{4(\tau - \bar{\tau})} \Pi^{+i} \Pi_i^- \quad (12)$$

To construct the Hilbert space of the gauge theory, we quantize the classical field configurations. For this, we adopt a functional Schrödinger polarization on the extended gauge theory configuration space $(A_+, A_-) \equiv (A - \frac{1}{\bar{\tau}} \tilde{A}, A - \frac{1}{\tau} \tilde{A})$ with the conjugate momenta $\Pi_i^\pm = -i \frac{\delta}{\delta A_\pm^i}$, so that the physical states are the wavefunctionals $\Psi[A_+, A_-] \equiv$

[§]Note that we could instead use the generator $\tilde{\mathcal{G}}$ in the following defined by varying the dual field component \tilde{A}_0 . This invariance under $A \leftrightarrow \tilde{A}$ applies to all of our constructions which follow, and it is the essence of the duality properties which we shall exhibit here.

$|A_+, A_- \rangle$. The oscillator form of the Hamiltonian (12) suggests to employ a coherent state quantization and take the wavefunctionals to be eigenstates of the momentum operators

$$\Pi_i^+(x)|A_+, A_- \rangle = \bar{\tau}\Lambda_i^+(x)|A_+, A_- \rangle \quad , \quad \Pi_i^-(x)|A_+, A_- \rangle = \tau\Lambda_i^-(x)|A_+, A_- \rangle \quad (13)$$

where Λ^\pm are one-forms on M_3 . These equations are solved by

$$|A_+, A_- \rangle = \exp \left[i \int_{M_3} d^3x \left(\bar{\tau}\Lambda_i^+(x)A_+^i(x) + \tau\Lambda_i^-(x)A_-^i(x) \right) \right] \quad (14)$$

The Gauss' law constraint is

$$\mathcal{G}(x)|A_+, A_- \rangle = \frac{1}{4\pi(\bar{\tau} - \tau)} \partial^i \left(\tau \frac{\delta}{\delta A_+^i(x)} - \bar{\tau} \frac{\delta}{\delta A_-^i(x)} \right) |A_+, A_- \rangle = 0 \quad (15)$$

which implies that $\Lambda_i^+(x) = \Lambda_i^-(x) + k_i^\perp(x)$, where k^\perp is a harmonic one-form on M_3 , $\partial_i k_i^\perp = 0$, representing the same degree of freedom as in (9). Thus the wavefunctionals, with momentum and harmonic quantum numbers Λ and k^\perp , are given by

$$|A_+, A_-; \Lambda, k^\perp \rangle = \exp \left[i \int_{M_3} d^3x \Lambda_i(x) \left(\bar{\tau}A_+^i(x) + \tau A_-^i(x) \right) + i \int_{M_3} d^3x k_i^\perp(x) A_+^i(x) \right] \quad (16)$$

The wavefunctionals (16) are eigenstates of the Hamiltonian (12) with the energy eigenvalues

$$\mathcal{E}_{\Lambda, k^\perp} = \int_{M_3} d^3x \frac{\tau}{4(\tau - \bar{\tau})} \Lambda^i(x) \left(\bar{\tau}\Lambda_i(x) + k_i^\perp(x) \right) \quad (17)$$

The Hilbert space \mathcal{H}_A spanned by the states (16) can be used to represent a unital \ast -algebra of observables appropriate to the gauge theory. The basic observables describing “interactions” in the theory are the gauge- and topologically-invariant Wilson line operators. In terms of the electric and magnetic charges q_e and q_m , the fundamental holonomy operators are

$$W_{q_e}^{(C)}[A] = \exp \left(iq_e \oint_C A \right) \quad , \quad \tilde{W}_{q_m}^{(C)}[\tilde{A}] = \exp \left(iq_m \oint_C \tilde{A} \right) \quad (18)$$

where $C \subset M$ are loops. Single-valuedness of these operators under large gauge transformations yields the Dirac quantization condition $q_e, q_m \in \mathbb{Z}$. The action of the Wilson loop operators on the Hilbert space can be computed by combining the two operators (18), using the Baker-Campbell-Hausdorff formula, into the abelian Wilson-'t Hooft line operators

$$W_{q_e, q_m}^{(C)}[A, \tilde{A}] = e^{-i\pi q_e q_m L(C)} W_{q_e}^{(C)}[A] \tilde{W}_{q_m}^{(C)}[\tilde{A}] = \exp \left[i \oint_C (q_+ A_+ + q_- A_-) \right] \quad (19)$$

where $L(C)$ is the self-linking number of the loop C and $q_+ = q_e + \tau q_m$, $q_- = q_e + \bar{\tau} q_m$. They describe the interaction of electric charges and magnetic monopoles with the electromagnetic field. We can write the contour integrals in (19) as

$$\oint_C A_\pm = \int_M d^4x A_\pm^\mu(x) J_\mu^{(C)\perp}(x) \quad (20)$$

where $J_\mu^{(C)\perp}(x) = \int_0^1 ds \dot{x}_\mu^{(C)}(s) \delta^{(4)}(x - x^{(C)}(s))$ is the conserved current of the closed worldline C , with $s \in [0, 1]$ the parameter of C and $x^{(C)}(s) : [0, 1] \rightarrow M$ the embedding of C in M . From this it follows that the action of the operators (19) on the wavefunctionals (16) yields

$$W_{q_e, q_m}^{(C)}[A, \tilde{A}] |A_+, A_-; \Lambda, k^\perp\rangle = |A_+, A_-; \Lambda(C), k^\perp(C)\rangle \quad (21)$$

where $\Lambda(C) = \Lambda + \frac{1}{\tau} q_- J^{(C)\perp}$ and $k^\perp(C) = k^\perp + [(1 + \frac{1}{\tau})q_+ - \frac{1}{\tau}q_-] J^{(C)\perp}$.

The algebra of the Wilson-'t Hooft operators (19) can also be computed. Assuming that the loop C is contractible (this is immediate if M is simply-connected) we can use Stokes' theorem to rewrite the line integrals of A_\pm as surface integrals of the curvatures F^\pm over a surface $\Sigma(C)$ spanned by C , i.e. $\partial\Sigma(C) = C$. A simple calculation shows that these operators then obey the clock algebra

$$W_{q_e, q_m}^{(C)}[A, \tilde{A}] W_{q'_e, q'_m}^{(C')}[A, \tilde{A}] = e^{q_+ L(C, C') q'_- / 2} W_{q'_e, q'_m}^{(C')}[A, \tilde{A}] W_{q_e, q_m}^{(C)}[A, \tilde{A}] \quad (22)$$

where

$$L(C, C') = \int_{\Sigma(C)} \int_{\Sigma(C')} [F^+, F^-] \quad (23)$$

defines a local intersection number of the curves C and C' . In fact, the representation of the operators (19) in terms of surfaces $\Sigma \subset M$, i.e. $W_{q^+, q^-}(\Sigma) = \exp[i \int_\Sigma (q^+ F^+ + q^- F^-)]$, emphasizes the fact that they are actually defined in terms of the elements (F^+, F^-) of the even, self-dual Lorentzian lattice $\Lambda = H^2(M; \mathbb{Z})$. For closed surfaces Σ , these operators depend only on the harmonic components p_\pm^\perp of the field strengths, and their clock algebra can be expressed in terms of the intersection matrix as

$$W_{q_a^+, q_a^-}(\Sigma^a) W_{q_b^+, q_b^-}(\Sigma^b) = e^{2\pi i q_a^+ G^{ab} q_b^-} W_{q_b^+, q_b^-}(\Sigma^b) W_{q_a^+, q_a^-}(\Sigma^a) \quad (24)$$

The Wilson-'t Hooft operators (19) form a basis for a noncommutative, infinite-dimensional unital $*$ -algebra \mathcal{A}_A . It describes topological and geometrical properties of the 4-manifold M , as well as the geometry of the complex line bundle $L \rightarrow M$. The construction of an algebra \mathcal{A}_A from an even self-dual lattice Λ was a crucial ingredient in the description of noncommutative string spacetimes in [8]. In that case, the spacetime was described by the vertex operator algebra of the underlying conformal field theory. Here we see that the algebra \mathcal{A}_A has a similar structure, in terms of both its definition and its algebraic relations. This sets a unified framework for the descriptions of both gauge theories and string theories using noncommutative geometry.

The algebra (24) is actually a large generalization of one of the original examples of a noncommutative geometry, the noncommutative torus [7, 15]. This algebra describes the quotient of the torus by the orbit of a free particle whose velocity forms an irrational angle with respect to the cycles of the torus. In this case the motion is ergodic and dense in the torus, and the resulting quotient is not a topological space in the usual sense. An equivalent way to visualize this is to consider the irrational rotations of a circle. It is then

possible to describe the space by the algebra of functions on a circle together with the action of these irrational rotations. Such an algebra is generated by two elements U and V which obey

$$UV = e^{i\pi\alpha} VU \quad (25)$$

with α the angle of rotation. This algebra has also appeared in the recent matrix model descriptions of M Theory [11, 13]. It is possible to prove [15] that the algebra generated by α and the ones generated by $\alpha + 1$ and $1/\alpha$ are equivalent. The algebra (24) is much larger than that of (25), and the role of the irrationality of α translates into a similar condition on τ . It is, however, very suggestive to notice that in our construction, for each choice of the charges q^\pm , the elements of the second homology group form a (larger dimensional) noncommutative torus.

The algebra \mathcal{A}_A is also intimately related to the loop space geometry of the manifold M . A generic algebra element is a linear combination given by the quantum mechanical path integral

$$W_{q_e, q_m}[A, \tilde{A}] = \int_{C^\infty(M, S^1)} [dx^{(C)}(s)] \widehat{\mathcal{W}}[x^{(C)}(s)] W_{q_e, q_m}^{(C)}[A, \tilde{A}] \quad (26)$$

over the loop space $C^\infty(M, S^1)$ of M , where $\widehat{\mathcal{W}}$ is a functional of only the worldline embeddings $x^{(C)}(s)$. Symbolically, we then have the diagram

$$\begin{array}{ccc} C^\infty(M, S^1) & \xrightarrow{W^{(\cdot)}} & \mathcal{A}_A \\ \pi \downarrow & & \downarrow \pi_{\mathcal{A}_A} \\ M & \xrightarrow{\chi} & C^\infty(M, \mathbb{C}) \end{array} \quad (27)$$

In this diagram the Wilson loops are regarded as functions on the loop space, which is viewed as an infinite-dimensional vector bundle over the manifold M with bundle projection π , where $\pi^{-1}(x)$ is the space of loops based at $x \in M$. The projection $\pi_{\mathcal{A}_A}$ is defined by restricting the Wilson loops to constant paths (i.e. loops of minimal area zero), which in turn projects the algebra \mathcal{A}_A onto the commutative algebra $C^\infty(M, \mathbb{C})$ describing the ordinary spacetime geometry of the 4-manifold M . We shall describe this projection somewhat more precisely below. For each $x \in M$, there corresponds the character $\chi_x(f) = f(x)$ of the algebra $C^\infty(M, \mathbb{C})$ which “reconstructs” the points, the topology, and the differentiable structure of the manifold M in purely algebraic terms [7]. It is in this sense that the noncommutative algebra \mathcal{A}_A is related to the usual geometry of four-dimensional electrodynamics and the manifold M , and moreover this point of view shows precisely what sort of geometry the noncommutative space here represents.

As discussed in [11], and as has been employed extensively in the case of string theory in [8]–[10] and for D-brane field theory in [14], the Dirac operator for the noncommutative space is obtained from the supercharges in an appropriate supersymmetrization of the bosonic field action. When the field theory is a two-dimensional sigma-model,

these operators project onto the Dirac-Ramond operators which describe the DeRham complex of the given target manifold [16]. In the case at hand, we consider the $N = 1$ supersymmetric abelian gauge theory with action

$$S[A, \psi, \bar{\psi}] = \frac{i}{4\pi} \int_M (F \wedge \tilde{F} + \text{Im}(\tau) \bar{\psi} \nabla \psi) \quad (28)$$

where ψ_α and $\bar{\psi}_{\dot{\alpha}}$, with $1 \leq \alpha, \dot{\alpha} \leq 4$, are fermion fields in the chiral and antichiral representations of $spin(4)$, respectively. For this model the $N = 1$ supersymmetry

$$\begin{aligned} \delta A_\mu &= i\bar{\varepsilon}_{\dot{\alpha}} \gamma_\mu^{\alpha\dot{\alpha}} \psi_\alpha + i\varepsilon_\alpha \gamma_\mu^{\alpha\dot{\alpha}} \bar{\psi}_{\dot{\alpha}} \\ \delta \psi_\alpha &= \bar{\varepsilon}_{\dot{\alpha}} \gamma_{\mu\nu}^{\alpha\dot{\alpha}} F^{\mu\nu} \quad , \quad \delta \bar{\psi}_{\dot{\alpha}} = \varepsilon_\alpha \gamma_{\mu\nu}^{\alpha\dot{\alpha}} F^{\mu\nu} \end{aligned} \quad (29)$$

with $\delta = \varepsilon_\alpha Q^\alpha + \bar{\varepsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}$, is generated by the action of the supercharges

$$Q^\alpha = \gamma_\mu^{\alpha\dot{\alpha}} \bar{\psi}_{\dot{\alpha}} \Pi^\mu \quad , \quad \bar{Q}^{\dot{\alpha}} = \gamma_\mu^{\alpha\dot{\alpha}} \psi_\alpha \Pi^\mu \quad (30)$$

on the quantum fields. If we define the generalized Dirac operator

$$\mathcal{D}^{\alpha\dot{\alpha}} = \gamma_\mu^{\alpha\dot{\alpha}} \otimes \Pi^\mu \quad (31)$$

then the supercharges (30) are generated by its action on the fermion fields, i.e. $Q^\alpha = \mathcal{D}^{\alpha\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}$ and $\bar{Q}^{\dot{\alpha}} = \mathcal{D}^{\alpha\dot{\alpha}} \psi_\alpha$.

However, there is another Dirac operator that can be introduced, namely that generated by the dual variables

$$\tilde{\mathcal{D}} = \gamma_\mu \otimes \tilde{\Pi}^\mu \quad (32)$$

This choice corresponds to representing the $N = 1$ supersymmetry above in terms of the dual gauge field configurations \tilde{A} . Correspondingly, we also introduce the self-dual and anti-selfdual Dirac operators

$$\mathcal{D}^\pm = \text{Im}(\tau)^{-1} \gamma_\mu \otimes \Pi_\pm^\mu \quad (33)$$

These Dirac operators all act on the Hilbert space $\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_A$. Accordingly, the algebra of Wilson-'t Hooft operators is augmented to $\mathcal{A} = C^\infty(M, \mathbb{C}) \otimes \mathcal{A}_A$. Since $\Pi_\pm^\mu \sim -i \frac{\delta}{\delta A_\mu^\mp}$ on the configuration space of the gauge theory, the operators (33) are indeed the appropriate Dirac operators for the present noncommutative geometry. Furthermore, they are related to the Hamiltonian operator (12) by

$$\mathbb{I} \otimes H = \int_{M_3} d^3x \frac{\text{Im}(\tau)}{4i} \mathcal{D}^+ \mathcal{D}^- \quad (34)$$

Thus, in the sense of (34), the “square” of the Dirac operators (33) coincides with the Hamiltonian, which in turn naturally defines the appropriate Laplace-Beltrami operator for the Riemannian geometry [9, 10].

The existence of two independent Dirac operators as metrics for the noncommutative space severely restricts the geometry. Namely, there exists a unitary transformation S :

$\mathcal{H} \rightarrow \mathcal{H}$ which is an automorphism of the algebra \mathcal{A} , i.e. $S\mathcal{A}S^{-1} = \mathcal{A}$, and maps the two Dirac operators (31) and (32) into one another as

$$S \not{D} S^{-1} = -\tilde{\not{D}} \quad , \quad S \tilde{\not{D}} S^{-1} = \not{D} \quad (35)$$

or equivalently $S \not{D}^+ S^{-1} = \tau \not{D}^+$, $S \not{D}^- S^{-1} = \bar{\tau} \not{D}^-$. This order-4 transformation is simply the S-duality transformation $\tau \rightarrow -1/\tau$ which amounts to an interchange of electric and magnetic degrees of freedom, $F \rightarrow \tilde{F}$ and $\tilde{F} \rightarrow -F$. It leaves both the action (7) and the Hamiltonian (34) invariant. Explicitly, it acts trivially on the spinor parts of \mathcal{H} and \mathcal{A} and on \mathcal{H}_A and \mathcal{A}_A it is defined by

$$S |A_+, A_-; \Lambda, k^\perp\rangle = |A_+, A_-; -\Lambda, \bar{\tau} k^\perp\rangle \quad , \quad S W_{q_e, q_m}[A, \tilde{A}] S^{-1} = W_{-q_m, q_e}[A, \tilde{A}] \quad (36)$$

In fact, the explicit operator S which implements this transformation is a gauge transformation of the noncommutative geometry, i.e. an inner automorphism of the algebra \mathcal{A}_A which acts as conjugation by the unitary element $S = e^{i\mathcal{F}} \in \mathcal{A}_A$, where[¶]

$$\mathcal{F} = \frac{1}{16\pi} \int_{M_3} d^3x \left(A_i \Pi^i - \tilde{A}_i \tilde{\Pi}^i \right) \quad (37)$$

The Hilbert space \mathcal{H} , the algebra \mathcal{A} , and the spectrum of the Hamiltonian H are invariant under the above transformation. In terms of the noncommutative geometry, this implies that the two spectral triples determined by the Dirac operators (31) and (32) are isomorphic,

$$(\mathcal{A} , \mathcal{H} , \not{D}) \cong (\mathcal{A} , \mathcal{H} , \tilde{\not{D}}) \quad (38)$$

As a change in choice of Dirac operator in the spectral triple is merely a change of metric from the point of view of the noncommutative geometry, the S-duality transformation is simply an isometry of the noncommutative space. It can also be viewed as a symmetry between an exterior derivative operator d and its adjoint $d^\dagger = \star d \star$ with respect to an appropriate \star -operator [8]. These features, as well as the fact that the S-duality transformation is a gauge symmetry, is the essence of duality in this formalism.

By construction, the basis wavefunctionals (16) of the Hilbert space are eigenstates of the Dirac operators (31) and (32). The noncommutative space is thus constructed in terms of the spectrum of the Dirac operators, which is a feature of the spectral action principle of noncommutative geometry [10, 17] which relates isospectral Riemannian geometries (ones with the same Dirac K-cycles (\mathcal{H}, D)) to one another. It is especially instructive to examine closely the zero mode eigenspaces $\ker \not{D}$ and $\ker \tilde{\not{D}}$. The isomorphism (38) when restricted to these subspaces of \mathcal{H} yields a physical interpretation of the

[¶]Strictly speaking, the unitary operator determined by (37) is only an element of the algebra of extended gauge-invariant observables determined by objects of the form (20) with arbitrary conserved currents J^μ . The continuity equations for the currents defined in (37) follow from the Maxwell equations of motion. Nevertheless, we take the form of the unitary transformation generated by (37) to mean that the S-duality transformation is an inner automorphism of \mathcal{A} .

duality symmetry. In $\ker \tilde{\mathcal{D}}$, the states, before the imposition of Gauss' law, are arbitrary functionals of A . On the other hand, the states of $\ker \mathcal{D}$ are arbitrary functionals of the dual gauge field \tilde{A} . This is the usual statement of S-duality when considered as a functional canonical transformation on the phase space of the gauge theory [6]. The pair (A, \tilde{A}) are in this sense a canonically conjugate pair of variables, and either one can be used as a configuration space coordinate depending on whether one works in the position or momentum representation.

Similar statements also hold for the algebra of Wilson-'t Hooft operators, where the zero-mode subspaces of \mathcal{A}_A are the commutants of the Dirac operators, i.e. the elements $W \in \mathcal{A}$ with $[\mathcal{D}, W] = 0$ or $[\tilde{\mathcal{D}}, W] = 0$. The mapping between commutants then corresponds to interchanging Wilson and 't Hooft line operators. This is the usual physical consequence of S-duality in gauge theory, i.e. it interchanges electric charges and magnetic monopoles, which has profound consequences for the non-perturbative structure of the quantum field theory [3]. Note that if one first imposes gauge invariance represented by Gauss' law and then projects \mathcal{A}_A onto these commutants, then the constraints become $J_\mu^{(C)\perp} = 0$, i.e. the Wilson loops which are defined on the constant loops $x^\mu(s) = x_0^\mu \in M \ \forall s \in [0, 1]$. The algebra \mathcal{A} thus projects onto $C^\infty(M, \mathbb{C})$, and this defines the projection $\pi_{\mathcal{A}_A}$ depicted in (27).

There is another “internal” symmetry of the quantum spacetime that is not represented as a change of Dirac operator but does lead to non-trivial dynamical effects in the quantum field theory. This is the symmetry under the shift $\theta \rightarrow \theta + 2\pi$ of the vacuum angle (or $\tau \rightarrow \tau + 1$). It corresponds to a change of the integer cohomology class of the instanton form $F \wedge F$, and in the present formulation it can be absorbed by the shift $\tilde{\Pi} \rightarrow \tilde{\Pi} + \Pi$, $\Pi \rightarrow \Pi$ of the momentum operators (and hence a redefinition of the Dirac operators). The corresponding spectral triples are each unaffected by these global redefinitions of all quantities. Note that these shifts correspond to reparametrizations of the elements $(\tilde{F}, F) \in \Lambda$ and the charges (q_m, q_e) which define the algebra \mathcal{A} . Thus only integer-valued multiples of 2π are allowed as vacuum angle shifts, in order that they leave invariant the algebra \mathcal{A}_A (equivalently to preserve the Dirac quantization condition). These two sets of modular transformations, $\tau \rightarrow -1/\tau$ and $\tau \rightarrow \tau + 1$, generate the duality group $SL(2, \mathbb{Z})$ of four-dimensional quantum electrodynamics under which τ transforms by linear fractional transformations and the vector $(\tilde{F}, F) \in \Lambda$ as a doublet. Recall that these were essentially also the symmetries of the algebra (25) of the noncommutative torus.

Thus the noncommutative geometry formulation of duality in gauge theories bears a remarkable resemblance to that of target space duality in string theory, just as other descriptions of duality seem to suggest [5, 6]. In particular, the gauge theory “spacetime” is described by an algebra \mathcal{A}_A which is determined by $b_2^+ b_2^-$ moduli parameters that parametrize the shape of the lattice Λ and take values in the Narain moduli space of Λ [8]

$$\mathcal{M}_{\text{qu}} = O(b_2^+, b_2^-; \mathbb{Z}) \setminus O(b_2^+) \times O(b_2^-) / O(b_2^+, b_2^-) \quad (39)$$

where the arithmetic group $O(b_2^+, b_2^-; \mathbb{Z}) = \text{Aut}(\Lambda)$ contains the group $\text{Diff}^+(M)$ of orientation preserving diffeomorphisms of the 4-manifold M . This is a subgroup of the group $\text{Diff}(M)$ of (outer) automorphisms of the algebra $C^\infty(M, \mathbb{C})$. The unity between the gauge field and string noncommutative geometries puts S-duality in string theory on the same footing as T-duality, as is required by M Theory [12]. It also agrees with the recent equivalences suggested by Matrix Theory [13] which unifies string theories within a gauge theoretical description. The algebraic similarities between \mathcal{A}_A and vertex operator algebras then suggest a representation of string scattering amplitudes in terms of Wilson line correlators. The unity of gauge fields and strings, and hence of the supersymmetric Yang-Mills theory that comprises Matrix Theory, appears to be contained in their relations to the noncommutative torus.

Acknowledgements: F.L. gratefully thanks the Theoretical Physics Department of Oxford University, where this work was begun, for hospitality. R.J.S. thanks N. Dorey and I. Kogan for insightful discussions on S-duality.

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